Thermodynamic properties of a one-dimensional system of charged bosons

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A large but finite one-dimensional neutral system containing two types of locally (and weakly) interacting "charged" bosons is examined for its thermodynamic behavior at finite temperatures. It is found that the system can have a length L_0 at which it will achieve thermodynamic stability provided the temperature is below a finite temperature T_d. If the temperature T_d is exceeded, the system disassociates in the sense that it no longer has a stable size. T_d is a function of the interaction strengths between the bosons as well as the number of bosons N present in the system. Although the system has an *a priori* dependence on a set of five parameters, when N and L are large scaling is present. Interestingly, if one of the interaction parameters is zero, making the interactions "Coulomb-like," the system will collapse if periodic boundary conditions are used. The introduction of Dirichlet boundary conditions does not prevent this collapse for large N and L but will do so otherwise. Moreover, without the collapse, the effect of N on the stability length is strikingly different from the nonzero parameter case, where periodic boundary conditions are used. This effect has been noted before in the ground-state energy, but now it is shown that the effect persists for finite temperatures.

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I. INTRODUCTION

In this paper we are going to examine some of the thermodynamic properties of a one-dimensional system of "charged" bosons obeying the Hamiltonian

$$H(c,g,N) = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\sum_{\substack{i=1\\ i<j}}^{N} (g + c e_i e_j)\delta(x_i - x_j) \qquad (1)$$

with

$$e_i = \begin{cases} +1 \text{ if } 1 \le i \le \dfrac{N}{2} \\ -1 \text{ if } \dfrac{N}{2}+1 \le i \le N \end{cases}$$

and $-\infty < x_i < \infty$. In the context of N identical bosons and with $c=0$ the solution for the ground-state energy has been known for some time now. Recently, it has been found that this class of Hamiltonian, with $c=0$, has emerged in some models dealing with two-dimensional (2D) random systems [1]. In addition, the form $H(c,O,N)$ has been used in work on directed Feynman paths with random phase [2]. Another approach taken to the same Feynman path problem [3] gave different results. This discrepancy led Blum, Koltun, and Shapir [4] to examine the Hamiltonian (1) when applied to a 1D quantum system consisting of an equal number of two types of charged bosons. To resolve the disagreement they directed their effort toward finding the exponent of N in the solution of the ground-state energy of the system. Using variational methods they were able to determine the N dependence of the ground-state energy for large values of N.

We [5] also applied (1) to the same quantum system used by Blum, Koltun, and Shapir. Our effort was aimed at finding an expression for the leading terms in the ground-state energy. We used a hybrid Bogoliubov perturbation method in the ground-state energy determination. Also, we were able to confirm the results of Blum, Koltun, and Shapir concerning the N dependence of the ground-state size when $0 \le g \ll c$ and with N large.

In this paper the Bogoliubov approach will be used again to see what can be determined about the thermodynamic properties of system (1) at finite temperatures and finite sizes. In Sec. II we will show the calculation method used. The interaction parameter space will be $c>0$ and $g \ge 0$. (The case with $c=0$ has already been dealt with by Yang and Yang [6].) The case of $g \ne 0$ will be treated separately from $g=0$ because of some essential differences. However, after the partition function is derived for each of the two g conditions, the calculation becomes very similar. Section III contains a discussion of the calculation results to illustrate the thermodynamic behavior of the system.

II. CALCULATIONS

A. The $g \ne 0$ case

Confining the system to a "box" of length L and going over to a Fock space constructed in terms of single-particle wave functions under periodic boundary conditions gives

$$H=\sum_{p=-\infty}^{\infty} p^2(a_p^\dagger a_p+b_p^\dagger b_p)+\left[\frac{g+c}{L}\right]\sum_{p=-\infty}^{\infty}(a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}a_{p_4}+b_{p_1}^\dagger b_{p_2}^\dagger b_{p_3}b_{p_4})\times\delta_{p_1+p_2;p_3+p_4}+2\left[\frac{g-c}{L}\right]\sum_p a_{p_1}^\dagger b_{p_2}^\dagger a_{p_3}b_{p_4}\delta_{p_1+p_2;p_3+p_4}\ . \tag{2}$$

Here, the $a_{p_n}, a_{p_n}^\dagger, b_{p_n}, b_{p_n}^\dagger$ are the annihilation and creation operator for the two types of bosons present. The periodic boundary conditions applied to the single-particle wave functions set the momentum $p=p_n=2\pi n/L$ where $n=\ldots,-2,-1,0,1,2,\ldots$. At this point the Bogoliubov approximation scheme is applied. The approximation is valid when the couplings c and g are weak and almost all particles are in the lowest-energy single particle, or $p=0$, state. Under these conditions the following approximations are valid:

$$a_0^\dagger\approx(N_0)^{1/2}=\left[\frac{N}{2}-\sum_{p\neq0}a_p^\dagger a_p\right]^{1/2}, \tag{3a}$$

$$a_0\approx(N_0)^{1/2}=\left[\frac{N}{2}-\sum_{p\neq0}a_p^\dagger a_p\right]^{1/2}, \tag{3b}$$

where N_0 is the occupation number of one of the types of bosons in the $p=0$ state. A similar approximation is made for the other type of boson present. The major contributions to (2) would come from terms involving the $p=0$ state and so all other terms will be dropped. Terms with only one zero-mode operator will also be dropped. In consequence, only terms producing numbers of $O(N)$ and larger are kept. The only terms left in (2) are now those involving four and two zero-mode operators. Conservation of momentum eliminates terms involving three zero-mode operators. In effect, following the prescription above, one sets

$$a_0=a_0^\dagger=\left[\frac{N}{2}-\sum_{p\neq0}a_p^\dagger a_p\right]^{1/2}, \tag{4a}$$

$$b_0=b_0^\dagger=\left[\frac{N}{2}-\sum_{p\neq0}b_p^\dagger b_p\right]^{1/2}, \tag{4b}$$

in (2) to get an approximation to the Hamiltonian. After some rearranging the result is

$$H\approx\frac{gN^2}{L}+\sum_{p\neq0}(a_p a_{-p}^\dagger b_p b_{-p}^\dagger)\begin{bmatrix} e'+e & 0 & e'-e & e'-e\\ p^2+2(e'+e) & e'+e & e'-e & e'-e\\ e'-e & e'-e & e'+e & 0\\ e'-e & e'-e & p^2+2(e'+e) & e'+e\end{bmatrix}\begin{bmatrix}a_{-p}\\ a_p^\dagger\\ b_{-p}\\ b_p^\dagger\end{bmatrix}, \tag{5}$$

where $e=cN/2L$ and $e'=gN/2L$. In (5) the gN^2/L originates from terms in (2) containing the four lowest, or $p=0$, mode operators. Introducing the operators $c_p^{(\pm)\dagger}=a_p^\dagger\pm b_p^\dagger/\sqrt{2}$ and $c_p^{(\pm)}=a_p\pm b_p/\sqrt{2}$ enables the carrying out of a block diagonalization of the Hamiltonian to

$$H\approx\frac{gN^2}{L}+\sum_{p\neq0}(c_p^{(+)}c_{-p}^{(+)\dagger}c_p^{(-)}c_{-p}^{(-)\dagger})\begin{bmatrix} 2e' & e'-e & 0 & 0\\ p^2+e+3e' & 2e' & 0 & 0\\ 0 & 0 & 2e & e-e'\\ 0 & 0 & p^2+3e+e' & 2e\end{bmatrix}\begin{bmatrix}c_{-p}^{(+)}\\ c_p^{(+)\dagger}\\ c_{-p}^{(-)}\\ c_p^{(-)\dagger}\end{bmatrix}. \tag{6}$$

Each of the two blocks on the diagonal is now "diagonalized" by introducing the Bogoliubov transformations

$$\begin{bmatrix}c_p^{(+)}\\ c_{-p}^{(+)\dagger}\end{bmatrix}=\begin{bmatrix}\cosh\theta^{(+)}(|p|) & -\sinh\theta^{(+)}(|p|)\\ -\sinh\theta^{(+)}(|p|) & \cosh\theta^{(+)}(|p|)\end{bmatrix}\begin{bmatrix}\beta_p\\ \beta_{-p}^\dagger\end{bmatrix},$$
$$\begin{bmatrix}c_p^{(-)}\\ c_{-p}^{(-)\dagger}\end{bmatrix}=\begin{bmatrix}\cosh\theta^{(-)}(|p|) & -\sinh\theta^{(-)}(|p|)\\ -\sinh\theta^{(-)}(|p|) & \cosh\theta^{(-)}(|p|)\end{bmatrix}\begin{bmatrix}\alpha_p\\ \alpha_{-p}^\dagger\end{bmatrix}, \tag{7}$$

and adjusting the value of $\theta^{(\pm)}(|p|)$ to carry out this requirement. The result of this formalism is

$$H\approx H_{\rm grd}+H_{\rm osc}\ , \tag{8a}$$

$$H_{\rm grd}=\frac{gN^2}{L}+\sum_{p\neq0}\{-p^2-2e-2e'+\tfrac{1}{2}(\mu_p+\omega_p)\}\ , \tag{8b}$$

$$H_{\rm osc}=\sum_{p\neq0}\{\mu_p\alpha_p^\dagger\alpha_p+\omega_p\beta_p^\dagger\beta_p\}\ , \tag{8c}$$

where $\mu_p\equiv\mu_n=|p|(p^2+8e)^{1/2}$ and $\omega_p\equiv\omega_n=|p|(p^2+8e')^{1/2}$. $H_{\rm grd}$ is the estimate of the ground-state energy.

As our aim is to study the thermodynamic properties of the system governed by H in (2) our next step is to determine the partition function Z defined as

$$Z(\beta,L)={\rm Tr}\{\exp(-\beta H)\}\ , \tag{9}$$

where $\beta=1/T$. Our method restricts us to the parameter region where the important contribution to (9) comes from those states in which most particles are in the single-particle ground state. In such a region we can use (8) in place of H in (9) to get

$$Z(\beta,L)=\exp(-\beta H_{grd})\mathrm{Tr}\{\exp(-\beta H_{osc})\} . \quad (10)$$

The trace sum in (10) is carried out over the quasiparticle states of (8) to obtain

$$Z(\beta,L) =\exp(-\beta H_{grd})\prod_{p\neq 0}\left[\frac{1}{1-\exp(-\beta\mu_p)}\frac{1}{1-\exp(-\beta\omega_p)}\right] . \quad (11)$$

Using the partition function we can calculate the thermodynamic measures of the system in a straightforward manner. The energy is given by

$$E=-\frac{\partial}{\partial\beta}\ln Z(\beta,L) =\frac{gN^2}{L}+2\sum_{n=1}^{\infty}[-p_n^2-2e-2e'+\tfrac{1}{2}(\mu_n+\omega_n)] +2\sum_{n=1}^{\infty}\left[\frac{\mu_n}{\exp(\beta\mu_n)-1}+\frac{\omega_n}{\exp(\beta\omega_n)-1}\right] . \quad (12)$$

Pressure is determined by using

$$P=\frac{1}{\beta}\frac{\partial}{\partial L}[\ln Z(\beta,L)]$$

to get

$$P=-\frac{\partial H_{grd}}{\partial L}+\frac{4}{L}\sum_{n=1}^{\infty}\left[\frac{\omega_n-2e'\frac{p_n^2}{\omega_n}}{\exp(\beta\omega_n)-1}+\frac{\mu_n-2e\frac{p_n^2}{\mu_n}}{\exp(\beta\mu_n)-1}\right] \quad (13a)$$

with

$$-\frac{\partial H_{grd}}{\partial L}=\frac{H_{grd}}{L}-\frac{2}{L}\sum_{n=1}^{\infty}p_n^2\left\{1-\frac{1}{4}\left[\frac{p_n^2}{\omega_n}+\frac{\omega_n}{p_n^2}+\frac{p_n^2}{\mu_n}+\frac{\mu_n}{p_n^2}\right]\right\} . \quad (13b)$$

The final determination using the partition function is for the entropy using $S=(\partial/\partial T)[T\ln Z(\beta,L)]$ to get

$$S=2\sum_{n=1}^{\infty}\left\{\ln\left[\frac{\exp(\beta\omega_n)}{\exp(\beta\omega_n)-1}\right]+\ln\left[\frac{\exp(\beta\mu_n)}{\exp(\beta\mu_n)-1}\right]+\frac{\beta\omega_n}{\exp(\beta\omega_n)-1}+\frac{\beta\mu_n}{\exp(\beta\mu_n)-1}\right\} . \quad (14)$$

Another of the system properties we consider is the specific heat which can be derived from (12) as

$$C_L=\left[\frac{\partial E}{\partial T}\right]_L =2\beta^2\sum_{n=1}^{\infty}\left\{\frac{\omega_n^2\exp(\beta\omega_n)}{[\exp(\beta\omega_n)-1]^2}+\frac{\mu_n^2\exp(\beta\mu_n)}{[\exp(\beta\mu_n)-1]^2}\right\} . \quad (15)$$

As already mentioned, in using the relationships derived above it must be borne in mind that the validity of the Bogoliubov method used is based on the assumption that the average number of particles in excited modes $\langle N_{ex}\rangle$ is small compared to the number of particles in the Bose condensate. Any calculation using a given set of values for the system parameters must be checked to ensure the system satisfies the ratio $R\equiv(\langle N_{ex}\rangle/N)\ll 1$. We can estimate the number of particles in excited modes through

$$\langle N_{ex}\rangle=\frac{\mathrm{Tr}\left\{\exp(-\beta H_{osc})\sum_{p\neq 0}(a_p^\dagger a_p+b_p^\dagger b_p)\right\}}{\mathrm{Tr}\{\exp(-\beta H_{osc})\}} =\sum_{n=1}^{\infty}\left\{-2+\left[\frac{\omega_n}{p_n^2}+\frac{p_n^2}{\omega_n}\right]\left[\frac{1}{2}+\frac{1}{\exp(\beta\omega_n)-1}\right]+\left[\frac{\mu_n}{p_n^2}+\frac{p_n^2}{\mu_n}\right]\left[\frac{1}{2}+\frac{1}{\exp(\beta\mu_n)-1}\right]\right\} . \quad (16)$$

It is of some interest at this point to examine the behavior of the quantities derived above when N and L are large. For sufficiently large N and L, the summation $\sum_{n=1}^{\infty}$ can be replaced with the continuous form $(L/2\pi)\int_0^{\infty}dp$ where $p_n=(2\pi n/L)\to p$. When (8b) is put in this continuous form the result is (with $N/L=\rho$)

$$H_{grd}=Nc\left\{\frac{g}{c}\rho-\frac{4}{3\pi}(c\rho)^{1/2}\left[1+\left[\frac{g}{c}\right]^{3/2}\right]\right\} . \quad (17)$$

From (11) comes, in the continuous form, the Helmholtz energy

$$F=-\frac{1}{\beta}\ln Z =Nc\left[\frac{g}{c}\rho+(c\rho)^{1/2}\left\{-\frac{4}{3\pi}\left[1+\left[\frac{g}{c}\right]^{3/2}\right]+\frac{1}{c\rho\beta}G\left[c\rho\beta,\frac{g}{c}\right]\right\}\right] . \quad (18)$$

Using (18) the continuous form of (12) can be written as

$$E=-\frac{\partial}{\partial\beta}\ln Z=-\frac{\partial}{\partial\beta}(-\beta F)=Nc\left[\frac{g}{c}\rho+(c\rho)^{1/2}\left\{-\frac{4}{3\pi}\left[1+\left[\frac{g}{c}\right]^{3/2}\right]+\frac{\partial}{\partial(c\rho\beta)}G\left[c\rho\beta,\frac{g}{c}\right]\right\}\right] . \quad (19)$$

Employing (18) and/or (19) gives the continuous forms of (13), (14), and (15), respectively, as

$$P=-\left[\frac{\partial F}{\partial L}\right]_T=g\rho^2+(c\rho)^{3/2}\left\{-\frac{2}{3\pi}\left[1+\left[\frac{g}{c}\right]\right]^{3/2}+(c\rho\beta)^{1/2}\frac{\partial}{\partial(c\rho\beta)}\left[\frac{1}{(c\rho\beta)^{1/2}}G\left[c\rho\beta,\frac{g}{c}\right]\right]\right\}, \tag{20}$$

$$S=\beta(E-F)=\frac{Nc}{(c\rho)^{1/2}}(c\rho\beta)^2\frac{\partial}{\partial(c\rho\beta)}\left[\frac{1}{c\rho\beta}G\left[c\rho\beta,\frac{g}{c}\right]\right], \tag{21}$$

$$C_L=\left[\frac{\partial E}{\partial T}\right]_L=\frac{Nc}{(c\rho)^{1/2}}(c\rho\beta)^2\frac{\partial^2}{\partial(c\rho\beta)^2}G\left[c\rho\beta,\frac{g}{c}\right]. \tag{22}$$

In (18)–(22) the function G is explicitly

$$G(z,y)=\frac{1}{\pi}\int_0^\infty dx\left\{\ln\left[1-\exp[-zx(x^2+4)^{1/2}]\right]+\ln\left[1-\exp[-zx(x^2+4y)^{1/2}]\right]\right\}. \tag{23}$$

Examination of the continuous equations above shows that all the quantities except for P are proportional to N and are thus extensive. While *a priori* any one of these thermodynamic quantities should depend on N, L, c, g, and T, in the continuous context there is a general scaling relationship of the function form

$$f(N,L,c,g,T)=N^d\hat{f}\left[\rho=\frac{N}{L},c,g,\frac{c\rho}{T}\right] \tag{24}$$

with

$$d=\begin{cases}1, & \text{for the extensive quantities}\\ 0, & \text{for the pressure.}\end{cases}$$

One striking feature of this scaling is that the temperature dependence enters via $c\rho/T$. Finally, it should be noted that trying to go to the continuous form of (16) yields a diverging integral when the lower limit is set to $p=0$. This divergence is a reflection of the fact that an infinite one-dimensional system does not allow a Bose condensation and without the condensate the Bogoliubov approximation is not valid. Although large systems may be considered our method does not support going to the thermodynamic limit $N,L\to\infty$ with N/L finite.

At this point we would like to add a comment on the ground-state energy as given in (17). Comparison of this formula with (18) given in [5] will show there is disagreement between the two. In [5] we used the approximation $a_0=a_0^\dagger=b_0=b_0^\dagger=(N/2)^{1/2}$ rather than the more accurate one given in (4). As this refinement only affects those terms containing four zero-mode operators, and basically results in the omission of a term of $O(N)$ from being added to a term of $O(N^2)$, the effect is not enough to change the qualitative results given in [5]. Indeed the quantitative results for the $g=0$ case are not affected at all. However, using (17) as the formula for the ground-state energy for periodic boundary conditions ($g>0$) is an improvement on that given in [5].

B. The $g=0$ case

If we set $g=0$ under periodic boundary conditions the system is unstable against collapse since the positive term gN^2/L in 8(b) is now absent. However, if Dirichlet boundary conditions rather than periodic ones are used then (2) becomes, as is shown in more detail in [5],

$$H=\sum_{n=1}^{\infty}p_n^2(a_n^\dagger a_n+b_n^\dagger b_n)+\frac{c}{2L}\sum_{m',n',m,n}f_{m'n'mn}(a_{m'}^\dagger a_{n'}^\dagger a_m a_n+b_{m'}^\dagger b_{n'}^\dagger b_m b_n-2a_{m'}^\dagger b_{n'}^\dagger a_m b_n), \tag{25a}$$

$$f_{m'n'mn}=\delta_{m'-n'+m-n}+\delta_{m'-n'-m+n}+\delta_{m'+n'-m-n}-\delta_{m'-n'+m+n}-\delta_{m'-n'-m-n}-\delta_{m'+n'+m-n}-\delta_{m'+n'-m+n}, \tag{25b}$$

where $p_n=\pi n/L$ ($n=1,2,3,\ldots$). After the Bogoliubov approximation is made the equation corresponding to (5) is

$$H\approx\frac{\pi^2N}{L^2}+\sum_{n=2}^{\infty}p_n^2(a_n^\dagger a_n+b_n^\dagger b_n)+e\sum_{n,m=2}^{\infty}f_{mn}\left[a_m^\dagger a_n+b_m^\dagger b_n+\frac{a_m^\dagger a_n^\dagger}{2}+\frac{b_m^\dagger b_n^\dagger}{2}+\frac{a_m a_n}{2}+\frac{b_m b_n}{2}-a_m^\dagger b_n^\dagger-a_m^\dagger b_n-b_m^\dagger a_n-a_m b_n\right], \tag{26a}$$

$$f_{mn}=2\delta_{m,n}-\delta_{m,n+2}-\delta_{m,n-2}, \tag{26b}$$

or in a more compact form, corresponding to (6),

$$H\approx\frac{\pi^2N}{L^2}+\sum_{n=2}^{\infty}\sum_{s=\pm}p_n^2c_n^{(s)\dagger}c_n^{(s)}+e\sum_{m,n=2}^{\infty}f_{mn}(2c_m^{(-)\dagger}c_n^{(-)}+c_m^{(-)\dagger}c_n^{(-)\dagger}+c_m^{(-)}c_n^{(-)}). \tag{27}$$

The term π^2N/L^2, that comes from the kinetic energies of the particles in the single-particle ground state is the term which provides the stability against collapse for the system. At this point in [5] we switched to the variational method and found that this amounted to the substitution $f_{mn}\to2\delta_{mn}$. From there we followed the Bogoliubov transformation method. The following observation serves as a support for this approximation. By using this approximation we arrived at the same term in the ground-state energy expression as we had determined in the periodic boundary case. That is, the terms in the Hamiltonian that are bilinear in operator content produced the same result regardless of boundary conditions suggesting that the couplings between modes n and $n\pm 2$ have no net contribution. We have applied the same approach here to (27).

Since one of the operator types in (27) is already present in the required diagonalized form it is only necessary to apply a Bogoliubov transformation to the other, namely,

$$\begin{bmatrix} c_n^{(-)} \\ c_n^{(-)\dagger} \end{bmatrix} = \begin{bmatrix} \cosh\theta_n & -\sinh\theta_n \\ -\sinh\theta_n & \cosh\theta_n \end{bmatrix} \begin{bmatrix} \alpha_n \\ \alpha_n^\dagger \end{bmatrix} . \tag{28}$$

This transformation leads to

$$H \approx H_{\text{grd}} + H_{\text{osc}} , \tag{29a}$$

$$H_{\text{grd}} = \frac{\pi^2 N}{L^2} + \tfrac{1}{2} \sum_{n=2}^{\infty} (\mu_n - p_n^2 - 4e) , \tag{29b}$$

$$H_{\text{osc}} = \sum_{n=2}^{\infty} (p_n^2 c_n^{(+)\dagger} c_n^{(+)} + \mu_n \alpha_n^\dagger \alpha_n) , \tag{29c}$$

which is the counterpart to (8). From here the calculations follow the same path as in the $g\neq 0$ case above giving, as listed below,

$$Z(\beta,L) = \exp(-\beta H_{\text{grd}}) \prod_{n=2}^{\infty} \left[\frac{1}{1-\exp(-\beta p_n^2)} \times \frac{1}{1-\exp(-\beta\mu_n)} \right] , \tag{30}$$

$$E = H_{\text{grd}} + \sum_{n=2}^{\infty} \left[\frac{p_n^2}{\exp(\beta p_n^2)-1} + \frac{\mu_n}{\exp(\beta\mu_n)-1} \right] , \tag{31}$$

$$P = -\frac{\partial H_{\text{grd}}}{\partial L} + \frac{2}{L} \sum_{n=2}^{\infty} \left[\frac{p_n^2}{\exp(\beta p_n^2)-1} + \frac{\mu_n - 2e\dfrac{p_n^2}{\mu_n}}{\exp(\beta\mu_n)-1} \right] , \tag{32a}$$

$$-\frac{\partial H_{\text{grd}}}{\partial L} = \frac{1}{L} H_{\text{grd}} + \frac{\pi^2 N}{L^3} - \frac{1}{4L} \sum_{n=2}^{\infty} p_n^2 \left\{ 2 - \left[\frac{\mu_n}{p_n^2} + \frac{p_n^2}{\mu_n} \right] \right\} , \tag{32b}$$

$$S = \sum_{n=2}^{\infty} \left\{ \ln\left[\frac{\exp(\beta p_n^2)}{\exp(\beta p_n^2)-1} \right] + \ln\left[\frac{\exp(\beta\mu_n)}{\exp(\beta\mu_n)-1} \right] + \frac{\beta p_n^2}{\exp(\beta p_n^2)-1} + \frac{\beta\mu_n}{\exp(\beta\mu_n)-1} \right\} , \tag{33}$$

$$C_L = \beta^2 \sum_{n=2}^{\infty} \left\{ \frac{p_n^4 \exp(\beta p_n^2)}{[\exp(\beta p_n^2)-1]^2} + \frac{\mu_n^2 \exp(\beta\mu_n)}{[\exp(\beta\mu_n)-1]^2} \right\} , \tag{34}$$

$$\langle N_{\text{ex}} \rangle = \sum_{n=2}^{\infty} \left\{ -\frac{1}{2} + \frac{1}{\exp(\beta p_n^2)-1} + \frac{1}{2} \left[\frac{p_n^2}{\mu_n} + \frac{\mu_n}{p_n^2} \right] \left[\frac{1}{2} + \frac{1}{\exp(\beta\mu_n)-1} \right] \right\} . \tag{35}$$

Taking the large N,L limit of (31)–(34) will produce the same results as found above in the $g\neq 0$ case (if one sets $g=0$, of course). So, not surprisingly, in the large N,L region where the continuous forms can be applied, the change in boundary conditions does not change the system's tendency to collapse. However, as will be brought out below, in the range of N and L where the term $\pi^2 N/L^2$ becomes significant not only does the system have a finite size but this size has a markedly different behavior than the $g>0$ case described above.

III. DISCUSSION

To illustrate some of the implications of the above derivations, we will look at a few of the general mathematical properties of an isotherm of the equation of state of the system. For small enough values of L the dominant term in the pressure determination is either gN^2/L^2 (for $g\neq 0$) or $\pi^2 N/L^3$ (for $g=0$). As a result, the pressure is positive for small enough L. An examination of (13) will show that for an isotherm at very large L the pressure effectively becomes

$$P = \frac{8}{L} \sum_{n=1}^{\infty} \frac{p_n^2}{\exp(\beta p_n^2)-1} \tag{36}$$

which in the continuous limit, with $L\to\infty$, is

$$P = \frac{4}{\pi} \int_0^{\infty} \frac{p^2 dp}{\exp(\beta p^2)-1} = \frac{\zeta\left[\frac{3}{2}\right](T)^{3/2}}{\pi^{1/2}} . \tag{37}$$

Turning off the interactions in (13) by setting $g=c=0$ will also produce this result. Thus, in the large L region the system behaves as a degenerate, noninteracting, Bose gas and (37) is the one-dimensional analog of the familiar result for such a gas as presented for example in [7]. Treating (32) in the same manner also leads to (37). Thus, for large L any isotherm of P vs L should asymptotically approach the positive value given in (37). On setting $T=0$ in (20) it is straightforward to show that $P=0$ when

$$L = \frac{9\pi^2 N}{4} \frac{g^2}{(c^{3/2}+g^{3/2})^2} .$$

The implication of the above is that for at least some range of $T>0$ the isotherms must cut the $P=0$ axis at two points.

Now we must add that the physical behavior of the actual system may not follow the mathematical form, outlined above, over the entire parameter space encompassed by the mathematics. As stated before, we must test to ensure we are within the range of validity of our approximation technique. As a consequence of this limitation, when $g > c$ the leading $g\rho^2$ term so dominates that any temperature variation in the isotherms is almost completely masked. It is only when $g < c$ that our method becomes useful in describing the thermodynamics of the system. Fortunately this parameter region may arguably be the more interesting for the following reasoning. The number of "attracting" pairs in the system is $N^2/4$ while the number of "repulsing" pairs is $(N^2/4)-(N/2)$. Examination of (1) will show that the "repulsions" are weighted as $g+c$ while the "attractions" are weighted as $g-c$. So with $g > c$ there are actually only repulsions present whereas in the $g < c$ to the $g \ll c$ region the system goes from predominantly repulsive to predominantly attractive.

To better illustrate some of these implications we will select a set of parameters and present the results in the form of numerically generated graphs. The numerical values given on the axes of the plots are in arbitrary units but these units apply to all the figures. For the purposes of establishing a limit below which the method is valid we have chosen to accept any calculation that leaves 95% or more of the particles in the Bose condensate. If this range is exceeded on any of the graphs then the point of violation is marked "limit" on the graph.

In each of the first four figures pressure and the Helmholtz free energy, $F=E-TS$, are plotted as functions of L for the values of the parameters g, c, N, and T given. Figure 1 displays the results for $T=0$. In Fig. 1 the pressure curve crosses the length axis at only one point but does approach this axis asymptotically for large values of L. If there is no external pressure constraint the system would assume the size L_0 where $P=0$. The minimum in the Helmholtz free-energy curve occurs as it should at L_0. As the Helmholtz free energy is equal to the energy at $T=0$, L_0 is the size of the ground-state system. In Fig. 2 the pressure curve now shows there are two values of L for which $P=0$. The minimum in the Helmholtz curve occurs as above at L_0. The other $P=0$ point corresponds to the maximum in the Helmholtz curve making this point an unstable system size. Figure 3 marks a kind of disassociation temperature which we will refer to as T_d. Below T_d the system localizes within a finite size as discussed above. Above T_d, as shown in Fig. 4, the pressure curve has no zero value and there are no "bound states." The Helmholtz curve has no minimum in Fig. 3 or Fig. 4 although it does have a point of inflection, as it should, in Fig. 3 at the L value where the P curve touches the horizontal axis. The pressure curve, in a descriptive sense, seems to be moved higher up the vertical scale as the temperature increases. All of the pressure curves given in the figures show a region where the inequality $(\partial P/\partial L)_T > 0$ holds. The system is, of course, thermodynamically unstable in this region.

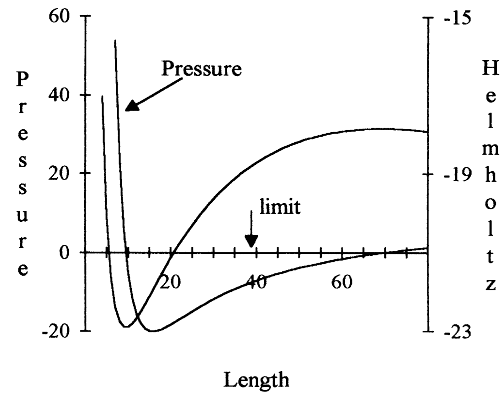

FIG. 2. Plot of pressure in units of 10^3 and Helmholtz free energy in units of 10^5 vs length for $T=400$. This temperature is below the disassociation temperature for $g=2\times10^{-5}$, $c=0.1$, and $N=10^6$. Physical quantities are in arbitrary units.

In Fig. 5 we display the specific heat versus temperature at three fixed lengths. All values used in these figures meet the criterion of $R < 0.05$. Under these conditions the specific heat increases monotonically with temperature. An examination of (15) and (34) shows that the specific heat is always positive.

The values of the parameters g, c, and N used in the graphs were chosen to show conveniently all the features

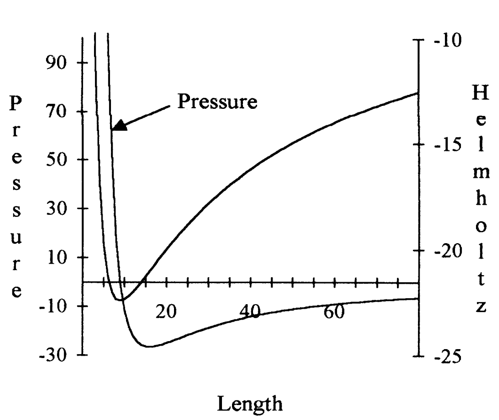

FIG. 1. Plot of pressure in units of 10^3 and Helmholtz free energy in units of 10^5 vs length for $T=0$. Here, $g=2\times10^{-5}$, $c=0.1$, and $N=10^6$. Physical quantities are in arbitrary units.

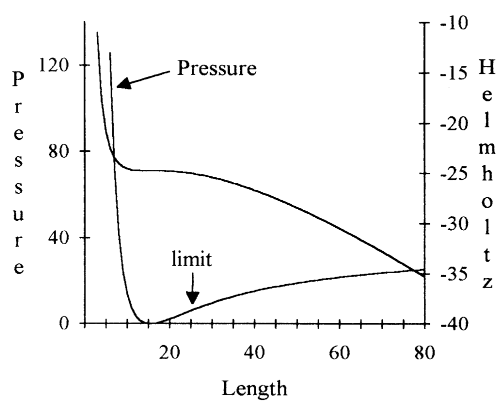

FIG. 3. Plot of pressure $\times10^3$ and Helmholtz free energy $\times10^5$ vs length for $T=974.8$. This temperature is the disassociation temperature for $g=2\times10^{-5}$, $c=0.1$, and $N=10^6$. Physical quantities are in arbitrary units.

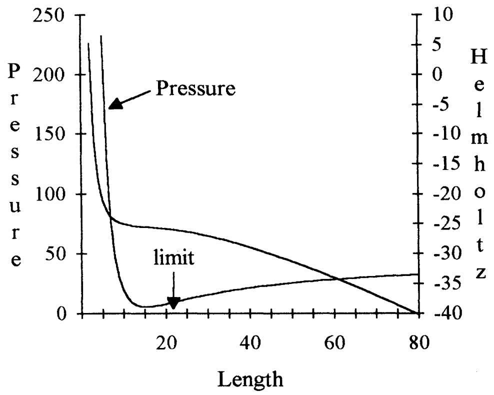

FIG. 4. Plot of pressure $\times 10^3$ and Helmholtz free energy $\times 10^5$ vs length for $T = 1100$. This temperature is above the disassociation temperature for $g = 2\times 10^{-5}$, $c = 0.1$, and $N = 10^6$. Physical quantities are in arbitrary units.

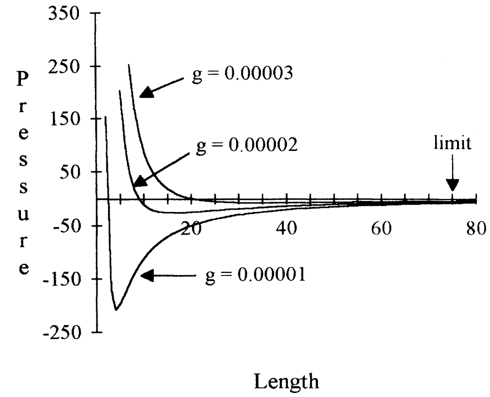

FIG. 6. Plot of pressure $\times 10^3$ vs length at a fixed temperature of 100. Here $c = 0.1$, $N = 10^6$, and the values of g are as indicated. Physical quantities are in arbitrary units.

of the thermodynamic behavior of the system under conditions where the approximation method used was valid. In Fig. 6 the $T = 100$ isotherm for $g = 1.0\times 10^{-5}$, $c = 0.1$, and $N = 10^6$ is plotted. Also potted are the results of doubling and tripling this value of g. Increasing the value of g increases the value of L_0. In this case increasing to $g = 6\times 10^{-5}$ would produce an L_0 that is outside our range of validity as determined by R. It can also be seen from the graph that an increase in g increases the minimum value reached by the pressure curve. This has the effect of decreasing the temperature T_d. In Fig. 7 curves for $T = 100$ are plotted but this time it is c that is doubled and tripled starting from $c = 0.05$. The effect is basically the reverse of the effect of increasing g.

The parameter space available to g and c is restricted to values that satisfy R less than some value felt to be reasonable by the user. If the system is to have an L_0 under this restriction then the requirement is $g \ll c$. This requirement was established for the ground-state energy in [5] and is here extrapolated to be required for finite temperatures. Our method will not allow us to determine whether or not the system has an L_0 if this restriction is not met.

As mentioned Sec. II B, reducing g to zero has the effect of collapsing the system. If periodic boundary conditions are replaced by Dirichlet ones, then the system collapse can be prevented but, as pointed out above, not for very large N and L. With the boundary condition change the qualitative behavior of the system for $g = 0$ is much the same as presented in the graphs already given. There is, however, one major difference in the variation of the value of L_0 as a function of N. Figure 8 illustrates this difference. For $g \neq 0$,

$$L_{p=0} \equiv L_0 = L_0(N,c,g,T) \approx N\frac{c}{T} f\left[cT^{-1/2}, \frac{g}{c}\right], \qquad (38)$$

as can be seen from (20). On the other hand, for the $g = 0$ case, L_0 does not depend on N in the simple manner given in (38). We will now show that the L_0 for $g = T = 0$ is a good approximation to the L_0 for $g = 0$ at any temperature well below the temperature T_d.

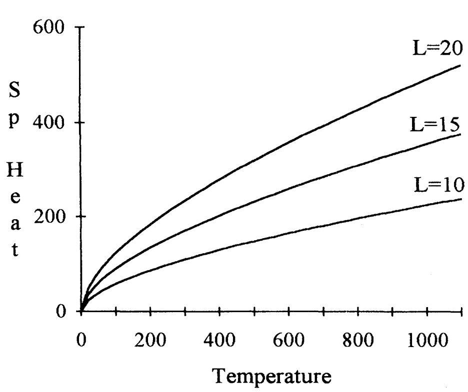

FIG. 5. Plot of specific heat vs temperature (at fixed length). These curves are for $g = 2\times 10^{-5}$, $c = 0.1$, $N = 10^6$, and L equal to the value indicated. Physical quantities are in arbitrary units.

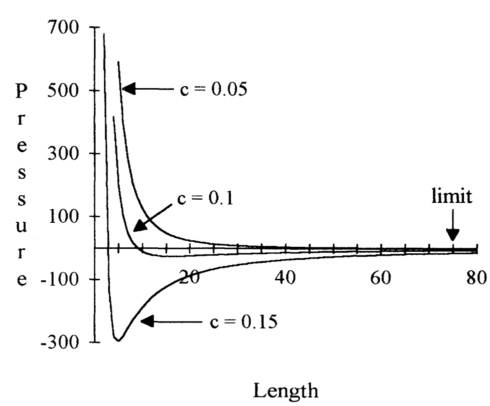

FIG. 7. Plot of pressure $\times 10^3$ vs length at a fixed temperature of 100. Here $g = 2\times 10^{-5}$, $N = 10^6$, and the values of c are as indicated. Physical quantities are in arbitrary units.

Taking the continuous form of (32) gives

$$P(L,T=0)=\frac{2\pi^2 N}{L^3}-\frac{2}{3\pi}\left[\frac{cN}{L}\right]^{3/2}[1+O((cNL)^{-1/2})] . \tag{39}$$

Solving (39) for L_0 sets

$$L_0(N,c,g=T=0) =\pi^2 3^{2/3}c^{-1}N^{-1/3}[1+O(c^{-1/2}N^{-1/3})] , \tag{40}$$

Here, $O(c^{-1/2}N^{-1/3})$ can safely be ignored for large N. In the vicinity of $L\approx L_0$, $e=cN/2L$ (which is proportional to $c^2N^{4/3}$) is very large. Under these conditions if β/L^2, which is proportional to $c^2\beta N^{2/3}$, is $\gg 1$ the temperature-dependent part of (32a) is of $O(c^3 N \exp[-4(3)^{-4/3}\pi^{-2}c^2\beta N^{2/3}])$ which makes it much smaller than the first term $O(c^3N^2)$, in (32a). Thus, the temperature dependence of L_0 is not very important if $T<c^2N^{2/3}$. Summarizing the above, we obtain as an estimate for L_0

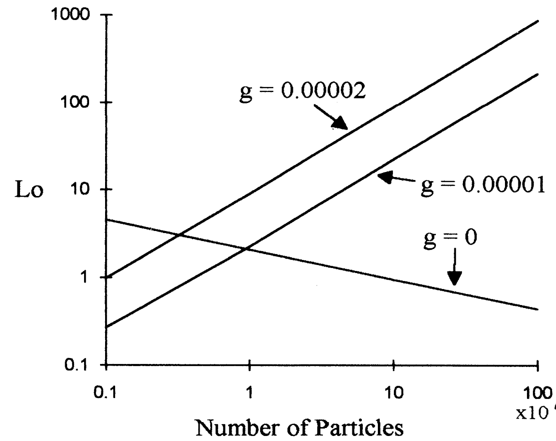

FIG. 8. Log-log plot of system size at $P=0$ as a function of the number of particles in the system. The parameters for all curves are set at $c=0.1$, and $T=100$ with the values of g as indicated in the figure. Physical quantities are in arbitrary units.

$$L_0(N,c,g=T=0)=\pi^2 3^{2/3}c^{-1}N^{-1/3}+O(c^3N^{-4/3}\exp[-4(3)^{-4/3}\pi^{-2}c^2\beta N^{2/3}]) . \tag{41}$$

So, for $g=0$, the system is the form of a "hard" core whose size is almost independent of T as long as T is well below the temperature T_d.

Figure 8 also shows that for some values of N the L_0 value for finite g can be less than that for $g=0$. Thus L_0 for $g=0$ cannot be extrapolated from the behavior of $g\to 0$. Again, the source of this seemingly peculiar behavior can be traced to the type of boundary conditions imposed. The comparison between $g=0$ and $g>0$ should be done under the same boundary conditions. In [5] we carried out the derivation of the ground-state energy for the $g>0$ case under Dirichlet boundary conditions. In that process we had to carry out a boson transformation that is described in the appendix of [5]. Unfortunately, with our use of the more rigorous approximation in (4) we have been unable to carry out the entire solution to the boson transformation so we cannot say what type of c-number term appears. Nevertheless, we can make the following argument. For both the $g=0$ and $g>0$ cases the Dirichlet boundary conditions generate the term π^2N/L^2. In the $g=0$ case it is the presence of this term that both prevents the system collapse and allows for $L_0\propto N^{-1/3}$ behavior. In the $g>0$ case the gN^2/L will usually be so much larger than π^2N/L^2 that the system has a $L_0\propto N$ behavior. However, if g is made small enough the point will be reached where the domination hierarchy of these two terms will change and the transition from the $L_0\propto N$ to the $L_0\propto N^{-1/3}$ behavior will be a continuous one and not the discontinuous process suggested by Fig. 8. We can, in fact, confirm this through the Boson transformation (cf. Appendix A of [5]) which can be explicitly performed for small gLN.

We will end this discussion section with one more note on scaling. As mentioned above, if the system is to have an L_0 that meets our method's restrictions, then the requirement is $g\ll c$. The magnitude of c can be related to N as follows. In Fig. 8, if one considers the plot for $g=0.00001$, for example, the R value for the point $N=10^5$ is approximately 0.00017. The R value increases as N is increased on this plot until at $N=10^8$ the R value has increased to just over 0.005. Obviously one cannot continue to increase N indefinitely with this set of parameters and if the system is to have a larger N then the value of c must be reduced. If one examines (16) it can be seen that for a given T and L a decrease in g and c by the same factor by which N is increased will leave $\langle N_{ex}\rangle$ unchanged. However, on the benefit side, R will be decreased by the same factor as the increase in N since $R=\langle N_{ex}\rangle/N$. Using this ploy it can be seen from (15) that the graphs of specific heat would be identical to the ones presented in Fig. 5 for any larger value of N provided g and c are both reduced as just explained. The ploy will not produce the same effect on the pressure curve because of the term gN^2/L^2. However, if c is reduced by the same factor by which N is increased and g is reduced so that the value of gN^2 is unchanged, then the pressure curves already presented can be made applicable for practical purposes to those for larger N. For example, changing the parameters in Fig. 2 to $N=10^{10}$, $c=10^{-5}$, $g=2\times 10^{-13}$ will produce a graph that is indistinguishable from Fig. 2 at the scale to which the figure is drawn. The R goes from $R=0.05$ at $L=37$ to $R=0.7\times 10^{-6}$ so that the pressure curve could now be extended to greater values of L and still remain within the domain of validity.

IV. CONCLUDING REMARKS

In this paper we have illustrated some of the thermodynamic properties of this one-dimensional system of bosons. The system would possess these attributes at least in the parameter space over which our approximation method is valid. The idea that the system has a natural or stable size was emphasized more heavily than other system properties. If the system is to have an L_0 then our method limits us to only be able to consider $0 \leq g \ll c$. The method will allow consideration of the parameter space $g > c$ but whether an L_0 exists under these conditions cannot be answered using this method. As the system temperature is increased from zero the size L_0 of the system increases. The temperature will eventually reach a "dissociation" value above which the system ceases to have a stable size and will expand without limit.

Particularly striking was the contrasting N dependence of L_0 between $g = 0$ and $g > 0$. This variation in N dependence was pointed out in [4] for the ground-state energy and we have shown that it persists at finite temperatures. However, one must be careful when $g = 0$ or is very close to zero as then the N dependence of L_0 is dependent on the boundary conditions used for the system. Finally, as was mentioned in the Introduction, other authors have interest in this system's size as they are working on problems for which this Hamiltonian is but a piece of the puzzle. It would be gratifying if our work as presented here was of some benefit to those writers.

ACKNOWLEDGMENTS

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